ANOMALY AND THERMODYNAMICS FOR 2D SPINORS IN THE AHARONOV–BOHM GAUGE FIELD

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ABSTRACT

The axial anomaly is computed for Euclidean Dirac fermions on the plane. The dependence upon the self-adjoint extensions of the Dirac operator is investigated and its relevance concerning the second virial coefficients of the anyon gas is discussed.

1. Spinors on the plane: A-B statistical interaction

The coupling of two dimensional fermions with the Aharonov–Bohm (A-B) gauge field leads to statistics transmutation and then to spinning anyon matter, whose special interesting features are worth being considered.

We first study the Euclidean Dirac operator on the plane

$$i\rlap{/}{D} = i\gamma_\mu(\partial_\mu - ieA_\mu) \quad , \tag{1.1}$$

the (A-B) gauge potential being $A_\mu \equiv \frac{\alpha}{e}\epsilon_{\mu\nu}\frac{x_\nu}{x_1^2+x_2^2}$, where $\gamma_1 = \sigma_1$, $\gamma_2 = \sigma_2$, $\gamma_3 = i\gamma_1\gamma_2 = \sigma_3$, *sigma*'s being the Pauli matrices, $\epsilon_{12} = 1$ and $-1 < \alpha < 0$. As is well known the field strength is $F_{\mu\nu} = -\frac{2\pi\alpha}{e}\epsilon_{\mu\nu}\delta^{(2)}(x)$.

Now let us consider the eigenvalues and eigenfunctions of the Dirac operator . The crucial point is that, in order to find a <u>complete</u> orthonormal basis which diagonalizes the Dirac operator it is necessary to consider its self-adjoint extensions [1][2]. To this concern let us choose polar coordinates (r, ϕ) on the plane and rescale the spinor wave function as

$$\frac{1}{\sqrt{\mu}}\psi_\lambda(r,\phi) \longmapsto \psi_{\lambda,n}(\xi,\phi) \equiv \begin{vmatrix} \psi_\lambda^{(L)}(\xi)e^{in\phi} \\ \psi_\lambda^{(R)}(\xi)e^{i(n+1)\phi} \end{vmatrix} \quad , \tag{1.2}$$

where μ is a suitable mass parameter to fix the scale of the eigenvalues (a natural choice is to set $\mu = e$), $n \in \mathbf{Z}$, $\lambda \in \mathbf{R} - \{0\}$ and $\xi = \mu r$. When $n \neq 0$ we get the eigenspinors regular at the origin: namely,

$$\psi_{\lambda,\pm n}(\xi,\phi) = \sqrt{\frac{|\lambda|}{4\pi}}\begin{vmatrix} (\pm i)J_{\pm\nu}(|\lambda|\xi)e^{\pm in\phi} \\ sgn(\lambda)J_{\pm(\nu+1)}(|\lambda|\xi)e^{i(1\pm n)\phi} \end{vmatrix} \quad ; \tag{1.3}$$

here $n \in \mathbf{N}$, J_ν being the Bessel function of order $\nu(\pm n) \equiv \pm n + \alpha$. On the other hand, the partial waves corresponding to $\nu(0) \equiv \alpha$ can not be both regular at the origin unless completeness of the eigenfunctions is lost [1]. Then one has to consider the self-adjoint extensions of the Dirac operator by means of the standard Von Neumann method of the deficiency indices. The corresponding eigenfunctions for $\nu = \alpha$ can be written in the form

$$\psi_{\lambda,0}^{(\omega)}(\xi,\phi) = \sqrt{\frac{|\lambda|}{4\pi(1+\sin\theta(|\lambda|)\cos\alpha\pi)}} \times$$

$$\left| \begin{array}{c} i\cos\frac{\theta(|\lambda|)}{2} J_\alpha(|\lambda|\xi) - i\sin\frac{\theta(|\lambda|)}{2} J_{-\alpha}(|\lambda|\xi) \\ sgn(\lambda)\left[\cos\frac{\theta(|\lambda|)}{2} J_{(1+\alpha)}(|\lambda|\xi) + \sin\frac{\theta(|\lambda|)}{2} J_{-(1+\alpha)}(|\lambda|\xi)\right]e^{i\phi} \end{array} \right| \quad , \tag{1.4}$$

where

$$\tan\theta(|\lambda|) = |\lambda|^{2\alpha+1}\tan\omega \quad . \tag{1.5}$$

The eigenfunctions in eq.s (1.3),(1.4) are improper eigenfunctions, since they belong to eigenvalues of the continuous spectrum. They are suitably normalized according to theory of the distributions, *viz.*

$$\lim_{R\to\infty} \int_0^{\mu R} \xi d\xi \int_0^{2\pi} d\phi\ \psi_{n_1}^\dagger(|\lambda_1|\xi,\phi)\psi_{n_2}(|\lambda_2|\xi,\phi) = \delta_{n_1 n_2}\delta(\lambda_1 - \lambda_2) \quad . \tag{1.6}$$

Moreover, in order to obtain the correct normalization as in eq. (1.6), one has to put the contribution at the origin equal to zero, thereby finding the relationship of eq. (1.5). For any value of ω, the purely continuous spectrum is the whole real line, due to the absence of zero modes.

2. The axial anomaly on the plane

Once the eigenvalue problem has been solved, we are able to set up the complex power by means of the spectral theorem. The complex power of the dimensionless operator $I_\omega^{-s} \equiv \left(\frac{\not{D}_\omega}{\mu}\right)^{-s}$ is defined by the kernel

$$< \xi_1,\phi_1|I_\omega^{-s}|\xi_2,\phi_2 > \equiv K_{-s}(I_\omega;\xi_1,\phi_1,\xi_2,\phi_2)$$

$$= 2\sum_{n=1}^\infty \left[\int_0^\infty d\lambda\ \lambda^{-s}\psi_{\lambda,n}(\xi_1,\phi_1)\psi_{\lambda,n}^\dagger(\xi_2,\phi_2) + (n\to -n)\right] \tag{2.1}$$

$$+ 2\int_0^\infty d\lambda\ \lambda^{-s}\psi_{\lambda,0}^{(\omega)}(\xi_1,\phi_1)\psi_{\lambda,0}^{(\omega)\dagger}(\xi_2,\phi_2) \quad ,$$

which can be analytically extended to a meromorphic function of the complex variable s; the key property is that the kernel of the complex power is regular at $s=0$. In particular, the value of its trace over spinor indices, on the diagonal $(\xi_1,\phi_1) = (\xi_2,\phi_2)$, can be explicitly evaluated [3] either in the case $\omega = 0,\pi, \quad -1 < \alpha < 0$, or in the case of semions: namely, $\alpha = -\frac{1}{2}, \omega \in \mathbf{R}$. We can properly construct the euclidean averages of the vector and axial currents, respectively, by means of point–splitting as well as analytic continuation [4], namely

$$< j_\mu^{(\omega)}(x) > = e < tr[\gamma_\mu\psi(x)\psi^\dagger(x)] > \equiv \lim_{s\to 1}\lim_{\epsilon\to 0} e\ tr[\gamma_\mu K_{-s}(I_\omega;x,x+\epsilon)] \tag{2.2a}$$

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where $< \cdot >$ means euclidean average and

$$< j_{\mu 3}^{(\omega)}(x) >= e < tr[\gamma_\mu \gamma_3 \psi(x)\psi^\dagger(x)] > \equiv \lim_{s \to 1} \lim_{\epsilon \to 0} e \ tr[\gamma_\mu \gamma_3 K_{-s}(I_\omega; x, x + \epsilon)] \quad (2.2b)$$

From the above definitions of the averaged local currents it is straightforward to show the quantum balance equations: namely, $< \partial_\mu j_\mu^{(\omega)}(x) >= 0$, testing the gauge invariance of the definition in eq. (2.2a), whereas

$$< \partial_\mu j_{\mu 3}^{(\omega)}(x) >= 2ie \lim_{s \to 0} tr[\gamma_3 \ K_{-s}(I_\omega; x, x)] \equiv \mathcal{A}^{(\omega)}(x) \qquad (2.3)$$

leads to the definition of the local axial anomaly, once the topology has been chosen in taking the limit $s \to 0$; we shall discuss below this delicate matter.

A first possibility is to consider the $\mathcal{S}'$-topology. If we take the limit $s \to 0$ in the sense of the distributions, it is straightforward to show that the limit exists only for $\omega = 0, \pi$ and we get

$$\int d^2x \ \mathcal{A}^{(0,\pi)}(x)f(x) = -\alpha \quad , \qquad (2.4)$$

where f is a suitable test function belonging to $\mathcal{S}(\mathbf{R}^2)$ normalized to $f(0) = \mu$; we notice that the above result, in full agreement with the one of Ref.[5], actually corresponds to the usual formula, $viz.$ $\mathcal{A}^{(0,\pi)}(x) = -\frac{ie^2}{2\pi}\epsilon_{\mu\nu}F_{\mu\nu}(x)$ as a distribution.

A second possibility is to consider the limit $s \to 0$ in the natural topology of $\mathbf{R} - \{0\}$ and <u>only afterwards</u> continue the result to $\mathcal{S}'(\mathbf{R}^2)$. As a matter of fact, a non vanishing result is obtained in this case for $\omega \neq 0, \pi$ and, when $\alpha = -\frac{1}{2}$, we can compute explicitely

$$\lim_{s \to 0} \lim_{\epsilon \to 0} e \ tr[\gamma_3 K_{-s}(I_\omega; x, x + \epsilon)]|_{\alpha=-\frac{1}{2}} = \frac{ie \sin \omega}{2\pi^2 r^2}, \quad r \neq 0 \quad . \qquad (2.5)$$

As a consequence, there is a unique continuation in $\mathcal{S}'(\mathbf{R}^2)$ which reads $(\alpha = -\frac{1}{2})$

$$\int_0^\infty \xi d\xi \int_0^{2\pi} d\phi \ f(\xi, \phi)\mathcal{A}^{(\omega)}(\xi) = \int_0^\infty \xi d\xi \int_0^{2\pi} d\phi \ \frac{ie \sin \omega}{2\pi^2[\xi^2]}f(\xi, \phi) \quad , \qquad (2.6)$$

where we recall the definition

$$\frac{1}{[\xi^2]} \equiv \frac{1}{2\mu^2}(\partial_1^2 + \partial_2^2)(\ln r)^2 + C(\mu)\delta^{(2)}(x) \quad , \qquad (2.7)$$

the arbitrary function $C(\mu)$ being there in order to guarantee the scaling property $\frac{1}{[\xi^2]} = \frac{1}{\mu^2} \cdot \frac{1}{[r^2]}$. We stress that, in the present case, if the test function vanishes at the origin, where the field strength is concentrated, still a nonvanishing contribution survives, of a purely quantum mechanical nature, which depends upon the parameter of the self–adjoint extensions, a quite interesting feature closely reminescent of the AB effect.

3. Thermodynamics and anomaly

The knowledge of the eigenvalues and eigenfunctions of $\not{D}_\omega$ allows to compute the

2nd virial coefficient for a gas of spinning anyons in 2+1 dimensions: namely,

$$a_2 = \mp \frac{\pi\beta}{2m}\left(1 \pm 4 \int d^2x \; [G_{int}(\beta; x, x) \pm G_{int}(\beta; x, -x)]\right) \quad, \tag{3.1}$$

where the upper and lower signs refer to bosons and fermions respectively, with

$$G_{int}(\beta; x, y) = tr < x|e^{-\beta H(\alpha)} - e^{-\beta H(0)}|y > \quad; \tag{3.2}$$

here $H(\alpha) = \slashed{D}^2_\omega$ is the 2-body spinning relative Hamiltonian while tr means trace over spinor indices. Since we are on the plane, eq. (3.1) needs some volume regularization and, in the present case, the suitable one is dimensional regularization to deal with products of regular and singular wavefunctions. The explicit calculation in the case $\omega = 0, \pi$ gives

$$a_2 = \mp \frac{\pi\beta}{2m}(1 \pm 2\alpha^2) \quad. \tag{3.3}$$

It turns out that the above result is different (for bosons) from the one of the standard spinless case [6] and, consequently, it appears that spin and singular wavefunctions indeed carry new features into the anyon physics. A further interesting point is the existence of a nice relationship between the 2nd virial coefficient and the axial anomaly. In the present case $\omega = 0, \pi$ one obtains

$$\frac{m}{2\pi\beta}[a_2^F(\alpha + 1) - a_2^B(\alpha)] = \int d^2x \; \mathcal{A}^{(0,\pi)}(x)f(x) \quad,$$

where $f(0) = m$. The corresponding relationships can be also found in the semion case for arbitrary values of ω. It would be very interesting to understand the properties of the N-anyon spinning gas and, in particular, the possible role of the singularities in the structure of the ground state.

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